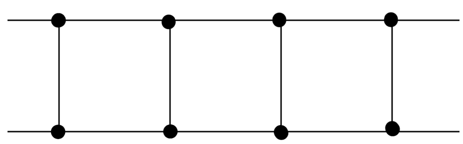
Resit exam Stochastic Processes

6 July 2021

- You have from 08.30 until 12.00. This includes the time needed to take pictures of your work and upload it to nestor dropbox.
- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- During the entire time you should be connected to the video call and be on camera, with sound turned on. Failure to do this will count as cheating.

Exercise 1 (20 pts)
We consider the simple random walk on the *infinite ladder*, the part of the integer lattice $\mathbb{Z}^2$ whose vertices are $\mathbb{Z} \times \{0, 1\}$. For definiteness, we start the random walk at the origin.



Is this random walk transient or recurrent? Justify your answer, clearly stating any results from the lectures or exercises you are using.

Exercise 2 (20 pts)
An urn contains n balls, some of which a red and the others blue. You repeatedly pick a ball uniformly at random, remove it from the urn and replace it with a ball *of the other colour*. Let R_t denote the number of red balls after we've done this t times. We can view $(R_t)_{t \geq 0}$ as a Markov chain with state space $S = \{0, \ldots, n\}$.

Determine all stationary distributions of the chain.
Clearly justify your answer, stating any results from the lecture notes or tutorial exercises you are using.

Exercise 3 (20 pts)
State and prove the Chernoff bound on the probability that a binomial random variable differs by more than λ from its mean.

(see next page)

Exercise 4 (20 pts)
The game "red now" is played using a standard deck of 52 cards, half of which are red. The cards are revealed one by one. You may say "red now" exactly once. If the next card to be revealed after you say that is red, then you win.

Let R_n denote the number of (not yet revealed) red cards left after n cards have been revealed. So deterministically $R_0 = 26$ and $R_{52} = 0$, while $R_1, \dots, R_{51}$ are random. We assume the ordering of the deck is completely random (uniform over all permutations of the 52 cards), and set

$$X_n := \frac{R_n}{52 - n}, \quad n = 0, 1, \dots, 51,$$

(i) Show that $(X_n)_{n=0,\dots,51}$ is a martingale (wrt. itself).

If you say "red now" at the beginning of the game, before any cards were revealed then you win with probability $\frac{1}{2}$. You might imagine that you can increase your chances of winning using some clever strategy, e.g. by waiting until the fraction of red cards left (that is to say, X_n) is high before you say "red now". As you will show next, in fact you cannot do better than just saying "red now" immediately.

(ii) Show that for all n, x : If n cards have been revealed and the fraction of red cards remaining is $X_n = x$, and you've not yet said "red now" then, under any possible strategy for the remaining moves, your chance of winning is no more than x.
(*Hint: use "reverse induction", i.e. induction on the number* $52 - n$ *of cards remaining. What happens in the base case when* $n = 51$*?*)

(iii) Explain how it follows that, no matter how clever a strategy you use, your chance of winning is at most $\frac{1}{2}$.
(A few well-chosen sentences will suffice.)

Exercise 5 (20 pts)
Let B denote standard Brownian motion, and let $0 < s < t$. Show that

$$\mathbb{P}(B(s) > 0, B(t) < 0) = \frac{\arccos\left(\sqrt{\frac{s}{t}}\right)}{2\pi}.$$

(Clearly state the facts about Brownian motion and normal distributions you are using.)

the end